A TENSOR PRODUCT FACTORIZATION FOR CERTAIN TILTING MODULES

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Abstract. Let G be a semisimple, simply connected linear algebraic group over an algebraically closed field k of characteristic $p > 0$. In a recent paper [6], Doty introduces the notion of $r-$minuscule weight and exhibits a tensor product factorization of a corresponding tilting module under the assumption $p \geq 2h-2$, where h is the Coxeter number. We remove this restriction and consider some variations involving the more general notion of $(p,r)-$minuscule weights.

Key Words: Tilting modules; Minuscule weights.

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Let G be a semisimple, simply connected linear algebraic group over an algebraically closed field k of characteristic $p > 0$. In a recent paper [6], Doty observed that the tensor product of the Steinberg module with a minuscule module is always indecomposable tilting. In this paper, we show that the tensor product of the Steinberg module with a module whose dominant weights are $p-$minuscule is a tilting module, not always indecomposable. We also give the decomposition of such a module into indecomposable tilting modules. Doty also proved that if $p \geq 2h-2$, then for $r-$minuscule weights the tilting module is isomorphic to a tensor product of two simple modules, usually in many ways. We remove the characteristic restriction on this result. A generalization of [4, proposition 5.5(i)] for $(p,r)-$ minuscule weights is also given. We start by setting up notation and stating some important definitions and results which will be useful later on.

Let $F : G \to G$ be the Frobenius morphism of G. Let B be a Borel subgroup of G and $T \subset B$ be a maximal torus of G. Let mod(G) be

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the category of finite dimensional rational $G-$modules. Define $X(T)$ to be the group of multiplication characters of T. For a $T-$module V and $\lambda \in X(T)$, write V^λ for corresponding weight space of V. Those λ's for which V^λ is non-zero are called weights of V. Any $G-$module M is completely reducible as a $T-$module. So M decomposes as a direct sum of its weight spaces and we have $M = \oplus_{\lambda \in X(T)} M^\lambda$ as a $T-$module. We will write $M^{[1]}$ for M^F. The Weyl group W acts on T in the usual way. Let $\{e(\lambda), \lambda \in X(T)\}$ be the canonical basis for the integral group ring $\mathbb{Z}X(T)$. The character of M is defined by $\operatorname{ch} M = \Sigma_{\lambda \in X(T)} (\dim M^\lambda)\, e(\lambda)$. For $\phi = \Sigma_{\mu \in X(T)} a_\mu e(\mu) \in \mathbb{Z}X(T)$ we set $\phi^{[1]} = \Sigma_{\mu \in X(T)} a_\mu e(p\mu) \in \mathbb{Z}X(T)$. Let Φ be the system of roots, Φ^+ the system of positive roots that make B the negative Borel. Let S be the set of simple roots. For $\alpha \in \Phi$ the co-root of α is denoted by α^v. Let $X^+(T)$ denotes the set of dominant weights. For $\lambda \in X^+(T)$, we define the set of $r-$restricted weights $X_r(T)$ by

$$X_r(T) = \{(\lambda, \alpha^v) < p^r : \text{for all simple roots } \alpha\}.$$

For $\lambda \in X^+(T)$, let $\Delta(\lambda)$ denote the Weyl module of highest weight λ, the dual Weyl module of highest weight λ is denoted by $\nabla(\lambda)$, and $L(\lambda)$ denotes the simple module of highest weight λ. The dual Weyl module $\nabla(\lambda)$ has simple socle $L(\lambda)$ and $\{L(\lambda) : \lambda \in X^+(T)\}$ is a complete set of pairwise non-isomorphic simple $G-$modules. A good filtration of a $G-$module M is defined as a filtration $0 = M_0 \leq M_1 \leq M_2 \leq ... \leq M_n = M$ such that for each $0 < i \leq n$, M_i/M_{i-1} is either zero or isomorphic to $\nabla(\lambda_i)$ for some $\lambda_i \in X^+(T)$.

A tilting module of G is a finite dimensional $G-$module M such that M and its dual module M^* both admit good filtrations. For each $\lambda \in X^+(T)$ there is an indecomposable tilting module $T(\lambda)$ which has highest weight λ. Every tilting module is a direct sum of copies of $T(\lambda)$, $\lambda \in X^+(T)$. For $\lambda \in X^+(T)$ the tilting module $T((p-1)\rho + \lambda)$ is projective as a G_1-module, where G_1 is the first infinitesimal group and ρ is the half sum of positive roots.

Now for $\lambda \in X_r(T)$, the modules $L(\lambda)$ form a complete set of pairwise non-isomorphic irreducible G_r modules. For $\mu \in X(T)$ let $\hat{Q}_r(\mu)$

denote the projective cover of $L(\mu)$ as a G_rT module see e.g [8] and [7]. The modules $\hat{Q}_r(\lambda)$, $\lambda \in X_r(T)$, form a complete set of pairwise non-isomorphic projective G_r modules. We refer the reader to [8], [4] and [2] for terminology and results not explained here.

A dominant weight λ is called minuscule if the weights of $\Delta(\lambda)$ form a single orbit under the action of W. Equivalently, by [1, chapter VIII, Section 7, proposition 6(iii)], λ is minuscule if $-1 \leq (\lambda, \alpha^v) \leq 1$ for all roots α. If $s(\lambda) = \Sigma_{\mu \in W\lambda}\, e(\mu)$ then λ minuscule implies $s(\lambda) = \operatorname{ch}\Delta(\lambda) = \operatorname{ch}\nabla(\lambda) = \operatorname{ch}L(\lambda)$. For $\lambda \in X^+(T)$ define λ to be $p-$minuscule if $\langle \lambda, \beta_0^v \rangle \leq p$, where β_0 is the highest short root. Moreover we define a weight $\lambda \in X_r(T)$ to be (p,r)-minuscule if $\lambda = \Sigma_{j=0}^{r-1}\, p^j \lambda^j$, where each λ^j is $p-$minuscule (and $\lambda^j \in X_1(T)$). In [6] Doty defines a weight λ to be $r-$minuscule if $\lambda = \Sigma_{j=0}^{r-1}\, p^j \lambda^j$, with each λ^j minuscule. Note that λ minuscule implies λ is $p-$minuscule. Similarly if λ is $r-$minuscule then λ is $(p,r)-$minuscule.

Definition. For $\lambda = \Sigma_{j=0}^{r-1}\, p^j \lambda^j \in X_r(T)$, $\lambda^j \in X_1(T)$ define

$$s_r(\lambda) = s(\lambda^0) s(p\lambda^1) \dots s(p^r \lambda^{r-1}).$$

Proposition 1. If λ is $(p,r)-$minuscule then

$$\operatorname{ch} T((p^r - 1)\rho + \lambda) = \chi((p^r - 1)\rho) s_r(\lambda).$$

Proof. By [4, theorem 5.3] we have if $\lambda \in X_1(T)$ and $T((p-1)\rho + \lambda)|_{G_1}$ is indecomposable then $T((p-1)\rho + \lambda) \otimes T(\mu)^{[1]} \simeq T((p-1)\rho + \lambda + p\mu)$ for all $\mu \in X^+(T)$. Also by the argument of [4, proposition 5.5] for $p-$minuscule (and restricted) λ we get that $T((p-1)\rho + \lambda)|_{G_1}$ is indecomposable. So we have $T((p^r - 1)\rho + \lambda) = \bigotimes_{j=0}^{r-1} T((p-1)\rho + \lambda^j)^{[j]}$. So $\operatorname{ch} T((p^r - 1)\rho + \lambda) = \Pi_{j=0}^{r-1} \operatorname{ch} T((p-1)\rho + \lambda^j)^{[j]}$. Since each λ^j is $p-$minuscule by [4, proposition 5.5] we get $\operatorname{ch} T((p-1)\rho + \lambda^j) = \chi((p-1)\rho) s(\lambda^j)$. Hence $\operatorname{ch} T((p^r - 1)\rho + \lambda) = \Pi_{j=0}^{r-1} \chi((p-1)\rho)^{[j]} s(\lambda^j)^{[j]}$. Combine this with above definition to get the result.

Remark. If λ is minuscule then $s(\lambda) = \operatorname{ch} L(\lambda)$ and hence $T((p-1)\rho+\lambda) = \mathrm{St} \otimes L(\lambda)$. Because these are tilting modules with same character. This gives us [6, lemma].

Lemma 1.

(a) if $\mu \in X^+(T)$ then $T((p^r-1)\rho) \otimes T(\mu)^{[r]} \simeq T((p^r-1)\rho + p^r\mu)$.

(b) suppose λ is minuscule then $\mathrm{St} \otimes L(\lambda) \simeq \hat{Q}_1((p-1)\rho + w_0\lambda)$ as G_1T modules, where w_0 is the longest element of W. In particular $\mathrm{St} \otimes L(\lambda)|_{G_1}$ is indecomposable.

(c) if λ is minuscule and $\mu \in X^+(T)$ then

$$\mathrm{St} \otimes L(\lambda) \otimes T(\mu)^{[r]} \simeq T((p-1)\rho + \lambda + p^r\mu).$$

Proof. By [8, II, 3.19] with $i = 0$ we have $\mathrm{St}_r \otimes \nabla(\mu)^{[r]} \simeq \nabla((p^r - 1)\rho + p^n\mu)$ for every $\mu \in X^+(T)$. It follows that $\mathrm{St}_r \otimes V^{[r]}$ is tilting for every tilting module V. In particular $\mathrm{St}_r \otimes T(\mu)^{[r]}$ is tilting. By [3, 2.1], $\mathrm{St}_r \otimes T(\mu)^{[r]}$ is isomorphic to $T((p^r-1)\rho+p^r\mu)$. This proves part (a).

Since

$$\begin{aligned}
&\operatorname{Hom}_{G_1T}(L((p-1)\rho + w_0\lambda), \mathrm{St} \otimes L(\lambda)) \\
&\qquad = \operatorname{Hom}_{G_1T}(L((p-1)\rho + w_0\lambda) \otimes L(\lambda)^*, \mathrm{St}) \\
&\qquad = \operatorname{Hom}_{G_1T}(L((p-1)\rho + w_0\lambda) \otimes L(-w_0\lambda), \mathrm{St}) \neq 0.
\end{aligned}$$

we have

$$\mathrm{St} \otimes L(\lambda)|_{G_1} = \hat{Q}_1((p-1)\rho + w_0\lambda) \oplus Z.$$

Also by [5, 1.2(2)], $\operatorname{ch} \hat{Q}_1((p-1)\rho + w_0\lambda) = \chi((p-1)\rho)\psi$, where $\psi = \Sigma a_\xi e(\xi)$ and $a_\xi \geq 0$ for all ξ.

Also by [8, II, 11.7, lemma(c)], $\operatorname{ch} \hat{Q}_1((p-1)\rho + w_0\lambda)$ is W invariant. This implies ψ is W invariant. Moreover $\hat{Q}_1((p-1)\rho+w_0\lambda)$ has unique highest weight $(p-1)\rho + \lambda$, so $\psi = s(\lambda) +$ lower terms. But ψ is W invariant and $\operatorname{ch} \hat{Q}_1((p-1)\rho + w_0\lambda)$ is divisible by $\chi((p-1)\rho)$ so we

must have $\psi = s(\lambda)$. So we get $Z = 0$ and $\operatorname{ch} \hat{Q}_1((p-1)\rho + w_0\lambda) = \operatorname{ch}(\mathrm{St} \otimes L(\lambda))$. This proves that

$$\mathrm{St} \otimes L(\lambda) \simeq \hat{Q}_1((p-1)\rho + w_0\lambda).$$

Now by [7, 4.2, Satz], $\hat{Q}_1((p-1)\rho + w_0\lambda)$ is indecomposable as G_1 module, so $\mathrm{St} \otimes L(\lambda)$ is indecomposable as G_1 module. Hence $\mathrm{St} \otimes L(\lambda)|_{G_1}$ is indecomposable. This proves part (b).

Since $\mathrm{St} \otimes L(\lambda)|_{G_1}$ is indecomposable by [3, 2.1] we get

$$\mathrm{St} \otimes L(\lambda) \otimes T(\mu)^{[r]} \simeq T((p-1)\rho + \lambda + p^r\mu).$$

This gives us result in part (c).

Proposition 2. Suppose λ is $r-$minuscule and $\mu \in X^+(T)$ then

$$\mathrm{St}_r \otimes L(\lambda) \otimes T(\mu)^{[r]} \simeq T((p^r-1)\rho + \lambda + p^r\mu).$$

Proof. Using Steinberg's tensor product theorem we get

$$\mathrm{St}_r \otimes L(\lambda) \simeq \bigotimes_{j=0}^{r-1} (\mathrm{St} \otimes L(\lambda^j))^{[j]}$$

where λ is $r-$minuscule. By above remark we have

$$\mathrm{St}_r \otimes L(\lambda) \simeq \bigotimes_{j=0}^{r-1} (T((p-1)\rho + \lambda^j))^{[j]}.$$

Apply lemma 1(c) inductively to get

$$\mathrm{St}_r \otimes L(\lambda) \simeq T((p^r-1)\rho + \lambda).$$

Now tensor both sides by $T(\mu)^{[r]}$ and apply lemma 1(c) again to get the result.

Corollary. Let λ is $r-$minuscule and $\mu \in X^+(T)$ then:

(a) $T((p^r-1)\rho + p^r\mu) \otimes L(\lambda) \simeq T((p^r-1)\rho + \lambda + p^r\mu)$.

(b) if $T(\mu)$ is simple then $\mathrm{St}_r \otimes L(p^r\mu + \lambda) \simeq T((p^r-1)\rho + p^r\mu + \lambda)$.

Proof. By lemma 1(a) we get $\mathrm{St}_r \otimes T(\mu)^{[r]} \simeq T((p^r-1)\rho + p^r\mu)$. Tensor this with $L(\lambda)$ to get the result in part (a).

If $T(\mu)$ is simple then $L(\mu) \simeq T(\mu)$. So $L(\lambda)\otimes T(\mu)^{[r]} \simeq L(\lambda)\otimes L(\mu)^{[r]}$. Using Steinberg's tensor product theorem we get $L(\lambda)\otimes L(\mu)^{[r]} \simeq L(\lambda + p^r\mu)$. Tensor this with $r-$th Steinberg module to get the result in part (b).

In case λ is $p-$minuscule it is of interest to determine the decomposition $\mathrm{St}\otimes \mathrm{L}(\lambda)$, $\mathrm{St}\otimes\Delta(\lambda)$ and $\mathrm{St}\otimes\nabla(\lambda)$ as a direct sum of indecomposable modules. In what follows we will show that these are all tilting modules and the direct sum decomposition is determined by the characters of $\nabla(\lambda)$ and $\mathrm{L}(\lambda)$. We will also show that if λ is $(p,r)-$minuscule then $\mathrm{St_r}\otimes\mathrm{L}(\lambda)$ is tilting. We will also give decomposition of $St_r\otimes\mathrm{L}(\lambda)$ into indecomposable tilting modules.

Lemma 2. Suppose λ is $p-$minuscule. Then every weight μ of $V(\lambda)$ satisfies $p\rho+\mu \in X^+(T)$, where $V(\lambda) = \Delta(\lambda)$ or $\nabla(\lambda)$.

Proof. If τ is a dominant weight of $V(\lambda)$ then τ is also $p-$minuscule because λ is the dominant weight so $\tau \leq \lambda$ and we can write $\lambda = \tau+\theta$ where θ is a sum of positive roots. Also $p \geq \langle\lambda, \beta_0^v\rangle = \langle\tau,\beta_0^v\rangle + \langle\theta,\beta_0^v\rangle \geq \langle\tau,\beta_0^v\rangle$.

Let μ be a weight of $V(\lambda)$ then $w\mu = \tau$ for some $p-$minuscule $\tau \in X^+(T)$ and $w \in W$. Let α be a simple root then $\langle p\rho+\mu, \alpha^v\rangle = p + \langle w^{-1}\tau, \alpha^v\rangle = p + \langle\tau, (w\alpha)^v\rangle$. So we need to show that $p+\langle\tau,\gamma^v\rangle \geq 0$ for all roots γ.

Now $p+\langle\tau,\gamma^v\rangle \geq 0$ for all roots $\gamma \iff p+\langle\tau,(w_0\gamma)^v\rangle \geq 0$ for all roots γ. And this is true $\iff p+\langle w_0\tau,\gamma^v\rangle \geq 0 \iff p - \langle -w_0\tau,\gamma^v\rangle \geq 0 \iff p-\langle\tau,\gamma^v\rangle \geq 0$. From the last inequality we get $\langle\tau,\gamma^v\rangle \leq p$ and since $\langle\tau,\gamma^v\rangle \leq \langle\tau,\beta_0^v\rangle \leq p$ we have the required result.

Recall that if $0 = M_0 \leq M_1 \leq ... \leq M_t = M$ is a chain of $B-$modules and $R\mathrm{ind}_B^G M_i/M_{i-1} = 0, 1 \leq i \leq t$ then for $\mathrm{ind}_B^G M$ we have a sequence $0 = \mathrm{ind}_B^G M_0 \leq \mathrm{ind}_B^G M_1 \leq ... \leq \mathrm{ind}_B^G M_t = \mathrm{ind}_B^G M$ with

$\text{ind}_B^G M_i/\text{ind}_B^G M_{i-1} \simeq \text{ind}_B^G M_i/M_{i-1}$. This follows by induction on t. Recall also that $R\text{ind}_B^G \mu = 0$ if $\langle \mu, \alpha^v \rangle \geq -1$ for all simple roots α. This follows by Kempf's vanishing theorem and [8, II, proposition 5.4(a)].

Proposition 3. Assume λ is $p-$minuscule and let V be a finite dimensional $G-$module such that $\mu \leq \lambda$ for all weights μ of V. Then $\text{St} \otimes V$ is a tilting module.

Proof. We will show that $\text{St} \otimes V$ has a $\nabla-$ filtration. Let μ be a weight of V, then μ is a weight of some composition factor $L(\nu)$ of V. Now $\nu \leq \lambda$, so $\langle \nu, \beta_0^v \rangle \leq \langle \lambda, \beta_0^v \rangle \leq p$, therefore ν is $p-$minuscule. Moreover μ is a weight of $L(\nu)$ and hence of $\nabla(\nu)$ and so by lemma 2 we have $p\rho + \mu \in X^+(T)$.

Now choose a $B-$module filtration of V given by $0 = V_0 \leq V_1 \leq ... \leq V_t = V$ with $V_i/V_{i-1} \simeq k_{\mu_i}$ where μ_i is a weight of V. Then $\text{St} \otimes V = \text{ind}_B^G((p-1)\rho \otimes V)$ and $(p-1)\rho \otimes V$ has a filtration $0 = (p-1)\rho \otimes V_0 \leq (p-1)\rho \otimes V_1 \leq ... \leq (p-1)\rho \otimes V_t = (p-1)\rho \otimes V$.

Also for each section $(p-1)\rho \otimes V_i/V_{i-1}$ we have $R\text{ind}_B^G((p-1)\rho \otimes V_i/V_{i-1}) = R\text{ind}_B^G((p-1)\rho \otimes k_{\mu_i}) = R\text{ind}_B^G((p-1)\rho + \mu_i) = 0$ because $\langle (p-1)\rho + \mu_i, \alpha^v \rangle \geq -1$. So $\text{St} \otimes V$ has filtration in section

$$\text{ind}_B^G((p-1)\rho \otimes V_i/V_{i-1}) = \begin{cases} \nabla(\mu_i), & \mu_i \in X^+(T) \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\text{St} \otimes V$ has a $\nabla-$filtration. Also $\mu^* \leq \lambda^*$ for all weights μ^* of V^* and λ^* is $p-$minuscule. So $\text{St} \otimes V^*$ has a $\nabla-$filtration. Therefore $(\text{St} \otimes V^*)^* = \text{St} \otimes V$ has a $\Delta-$filtration. Hence $\text{St} \otimes V$ is tilting.

Corollary. Suppose λ is $p-$minuscule then $\text{St} \otimes \Delta(\lambda) \simeq \text{St} \otimes \nabla(\lambda)$.

Proof. By proposition 3, $\text{St} \otimes \Delta(\lambda)$ and $\text{St} \otimes \nabla(\lambda)$ are tilting modules. Moreover $\text{St} \otimes \Delta(\lambda)$ and $\text{St} \otimes \nabla(\lambda)$ have the same character and hence are isomorphic.

Theorem 1. Let λ is $p-$minuscule and V be a finite dimensional $G-$module such that $\mu \leq \lambda$ for all weights μ of V. Then

$$\mathrm{St} \otimes V \simeq \bigoplus_{\nu \in X^+(T)} a_\nu\, T((p-1)\rho + \nu)$$

where $\mathrm{ch}(V) = \Sigma_{\nu \in X^+(T)}\, a_\nu s(\nu)$.

Proof. By proposition 3 we have $\mathrm{St} \otimes V$ is a tilting module. Also by [4, proposition 5.5] we get $\mathrm{ch}\, T((p-1)\rho + \nu) = \chi((p-1)\rho)s(\nu)$. Write $\mathrm{ch}\,(V) = \Sigma_{\nu \in X^+(T)}\, a_\nu s(\nu)$ then the tilting modules $\mathrm{St} \otimes V$ and $\oplus_{\nu \in X^+(T)}\, a_\nu T((p-1)\rho + \nu)$ have the same character and hence are isomorphic.

Proposition 4. Assume λ is $(p, r)-$minuscule then $\mathrm{St_r} \otimes L(\lambda)$ is a tilting module.

Proof. Since λ is $(p, r)-$minuscule this implies $\lambda \in X_r(T)$ and $\lambda = \Sigma_{j=0}^{r-1}\, p^j \lambda^j$, where each λ^j is $p-$minuscule. Using Steinberg tensor product theorem we have $\mathrm{St_r} \otimes L(\lambda) = \bigotimes_{j=0}^{r-1} (\mathrm{St} \otimes L(\lambda^j))^{[j]}$. By proposition 3, $\mathrm{St} \otimes L(\lambda^j)$ is tilting for each λ^j. We will use mathematical induction to complete the proof.

Write $\mathrm{St_r} \otimes L(\lambda) = \mathrm{St} \otimes L(\lambda^0) \otimes (\mathrm{St} \otimes L(\lambda^1) \otimes \mathrm{St}^{[1]} \otimes L(\lambda^2)^{[1]} \otimes ... \otimes \mathrm{St}^{[r-2]} \otimes L(\lambda^{r-1})^{[r-2]})^{[1]}$. Then using inductive hypothesis and theorem 1 we have $\mathrm{St_r} \otimes L(\lambda) = \bigoplus_\mu a_\mu \mathrm{St} \otimes L(\lambda^0) \otimes T(\mu)^{[1]}$. Also by theorem 1, $\mathrm{St} \otimes L(\lambda^0) = \bigoplus_{\nu \in X^+(T)} b_\nu\, T((p-1)\rho + \nu)$. So $\mathrm{St_r} \otimes L(\lambda) = \bigoplus_{\mu,\nu} a_\mu b_\nu\, T((p-1)\rho + \nu) \otimes T(\mu)^{[1]}$. Hence $\mathrm{St_r} \otimes L(\lambda)$ is tilting.

Theorem 2. Let λ is $(p, r)-$minuscule then

$$\mathrm{St_r} \otimes L(\lambda) \simeq \bigoplus_{\nu \in X^+(T)} b_\nu\, T((p^r - 1)\rho + \nu)$$

where $\mathrm{ch}\, L(\lambda) = \Sigma_{\nu \in X^+(T)}\, b_\nu s_r(\nu)$.

Proof. $\mathrm{St_r} \otimes L(\lambda)$ is tilting by proposition 4. Also by proposition 1 we have $\mathrm{ch}\, T((p^r - 1)\rho + \nu) = \chi((p^r - 1)\rho)s_r(\nu)$. Write $\mathrm{ch}\, L(\lambda) = \Sigma_{\nu \in X^+(T)}\, b_\nu s_r(\nu)$ then the tilting modules $\mathrm{St}_r \otimes L(\lambda)$ and

$\bigoplus_{\nu \in X^+(T)} b_\nu T((p^r - 1)\rho + \nu)$ have the same character and hence are isomorphic.

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